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Inflationary Brans–Dicke Universe and G

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In a recent paper, Mathiazhagan and Johri reduced the field equations for an isotropic, homogeneous, and almost flat universe with a constant vacuum-energy density by Brans–Dicke theory to a pair of coupled differential equations. They also obtained a particular solution of these equations. Further, they used this particular solution of the equations to estimate the value of the gravitational constant. Here we obtain the complete set of solutions of the above-mentioned coupled differential equations and improved the estimate of Mathiazhagan and Johri of the gravitational constant.

1. INTRODUCTION

In a recent paper, Mathiazhagan and Johri (1984) obtained the following set of equations for an isotropic, homogeneous, and almost flat universe ($k=0$) with a constant vacuum-energy density in Brans–Dicke theory:

$$\frac{\dot{R}^2}{R^2}=\frac{8\pi\rho_v}{3\phi}-\frac{\dot{\phi}\dot{R}}{\phi R}+\frac{\omega\dot{\phi}^2}{6\phi^2}$$
$$\ddot{\phi}+\frac{3\dot{R}\dot{\phi}}{R}=\frac{32\pi\rho_v}{3+2\omega} \tag{1.1}$$

where the energy-momentum tensor is

$$T_{ij}^{\text{vac}}=g_{ij}\rho_v \tag{1.2}$$

ρ_v is the vacuum-energy density, $R(t)$ is the scale factor of the universe, ϕ is the scalar field of the Brans–Dicke (1961) equation, and ω is the dimensionless Brans–Dicke coupling constant.

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Mathiazhagan and Johri (1984) obtained a particular solution of equations (1.1) given by

$$R = \left(\frac{t}{t_0}\right)^{\omega+1/2}$$
$$\phi = \frac{32\pi\rho_v}{(3+2\omega)(5+6\omega)}t^2 \tag{1.3}$$

where t_0 is an arbitrary constant.

They used this particular solution (1.3) to estimate the value of the gravitational constant. In the present work, we obtain the complete set of solutions of equations (1.1). We try also to improve their work to estimate the value of the gravitational constant.

2. SOLUTIONS

Equations (1.1) can be reduced to

$$\dot{\phi} = n\phi\left[\frac{\phi^2 R_\phi^2}{R^2} + \frac{\phi R_\phi}{R} - l\right]^{-1}$$
$$\ddot{\phi} = m - 3n\frac{\phi R_\phi}{R}\left[\frac{\phi^2 R_\phi^2}{R^2} + \frac{\phi R_\phi}{R} - l\right]^{-1} \tag{2.1}$$

where

$$m = \frac{32\pi\rho_v}{3+2\omega}, \qquad n = \frac{8\pi\rho_v}{3}, \qquad l = \frac{\omega}{6}, \qquad R_\phi = \frac{dR}{d\phi} \tag{2.2}$$

Using $\ddot{\phi} = \frac{1}{2}\, d\dot{\phi}^2/d\phi$, we can rewrite equations (2.1) as

$$y\left(y^2 - \frac{p}{12}\right)\left(y - \frac{p}{4}\right) + \frac{p}{12}y_x = 0 \tag{2.3}$$

where

$$p = 3+2\omega, \qquad y = \frac{d}{dx}(\ln R) + \tfrac{1}{2}$$

$$x = \ln|\phi|, \qquad y_x = \frac{dy}{dx}$$

Equations (2.3) trivially gave the following solutions.

Solution I. $y=0$. In this case, the solutions of equations (1.1) are given by

$$
\begin{aligned}
R &= \pm \frac{c_1}{\{c_2 \pm [8\pi\rho_v/(3+2\omega)]^{1/2} t\}^2} \\
\phi &= -\left[c_2 \pm \left(\frac{8\pi\rho_v}{3+2\omega}\right)^{1/2} t\right]^2
\end{aligned}
\tag{2.4}
$$

where c_1, c_2 are constants of integration.

Solution II. $y=p/4$. In this case, the solutions of equations (1.1) are given by

$$
\begin{aligned}
R &= c_3\left[\frac{32\pi\rho_v}{(3+2\omega)(5+6\omega)} t^2 + At + B\right]^{(\omega+1/2)/2} \\
\phi &= \frac{32\pi\rho_v}{(3+2\omega)(5+6\omega)} t^2 + At + B
\end{aligned}
\tag{2.5}
$$

where c_3, A, and B are constants of integration.

If one puts $A=0=B$, and

$$
c_3\left[\frac{32\pi\rho_v}{(3+2\omega)(5+6\omega)}\right]^{(\omega+1/2)/2} = \left(\frac{1}{t_0}\right)^{\omega+1/2}
$$

then one can get Mathiazhagan and Johri's (1984) particular solution given by (1.3).

Solution III. $y=\pm(p/12)^{1/2}$. In this case, the solutions of equations (1.1) are given by

$$
\begin{aligned}
R &= c_4\left[\frac{16\pi\rho_v}{3+2\omega} t^2 + A't + B'\right]^{-1/2\pm[(3+2\omega)/12]^{1/2}} \\
\phi &= \frac{16\pi\rho_v}{3+2\omega} t^2 + A't + B'
\end{aligned}
\tag{2.6}
$$

where c_4, A', and B' are constants of integration.

Solution IV. $y \neq \text{const}$, $\omega \neq -5/6$. In this case, the solutions of equations (1.1) are given by

$$
\begin{aligned}
c_5 R &= \frac{(y-a)^{(2a-1)/[4(3a-1)]}(y+a)^{(2a+1)/[4(3a+1)]}}{(y-3a^2)^{(6a^2-1)/[2(9a^2-1)]}} \\
c_6|\phi| &= \frac{(y-a)^{1/[2(3a-1)]}}{(y+a)^{1/[2(3a+1)]}(y-3a^2)^{1/(9a^2-1)}}
\end{aligned}
\tag{2.7}
$$

where c_5 and c_6 are constants of integration and $a^2=p/12=(3+2\omega)/12$.

Solution V. $y \neq \text{const}$, $\omega = -5/6$. In this case, the solutions of equations (1.1) are given by

$$c_7 R = \left[\left(\frac{3y-1}{3y+1}\right)^{5/4} e^{1/[2(3y-1)]}\right]^{-1/6}$$
$$c_8|\phi| = \left(\frac{3y-1}{3y+1}\right)^{1/4} e^{1[2(3y-1)]} \tag{2.8}$$

where c_7 and c_8 are constants of integration.

Thus, we have obtained the complete set of solutions of equations (1.1).

3. ESTIMATION OF THE VALUE OF THE GRAVITATIONAL CONSTANT BY CONSIDERING MATHIAZHAGAN AND JOHRI'S SOLUTION

To determine theoretically the present value of the gravitational constant, we shall follow the revised version of the inflationary universe scenario (Linde, 1982; Albrecht and Steinhardt, 1982) within the framework of Brans–Dicke theory. We take an $SU(5)$ model with Coleman–Weinberg (CW) symmetry-breaking mechanism. The effective zero-temperature ($T = 0$) CW scalar potential is

$$V_0(\phi) = B\phi^4\left(\ln\frac{\phi^2}{\sigma^2} - \frac{1}{2}\right) + \tfrac{1}{2}B\sigma^4 \tag{3.1}$$

where ϕ is related to the adjoint Higgs field, $B = (5625/1024\pi^2)g^4$, g^2 is the gauge coupling parameter, and $\sigma = 4.5 \times 10^{14}$ GeV. A temperature-dependent term V_T should be included in the above potential for a finite temperature. When $T \ll M_x$ $[M_x^2 = (25/8)g^2\sigma^2]$, the effective scalar potential, including the temperature correction, in the range $0 < \phi \leq \sigma$ is [following Abbot (1981) and Sher (1982)]

$$V(\phi) = -\frac{5625g^4}{512\pi^2}\phi^4 \ln\left(\frac{M_x}{T}\right) + \frac{75}{16}g^2T^2\phi^2 + \tfrac{1}{2}B\sigma^4 \tag{3.2}$$

We know that when the temperature of the universe is greater than the critical temperature ($T_{\text{Gut}} \approx \sigma$) the symmetric ($\phi = 0$) vacuum is favored. As the temperature of the universe approaches the critical temperature, a second minimum develops at $\phi \approx \sigma$. Once the temperature of the universe drops below the critical temperature ($T_{\text{Gut}} \approx \sigma$) in the course of expansion, the symmetric phase $\phi \approx 0$ still remains as a local minimum because a potential barrier at $\phi \approx T$ forbids it to make a transition from $\phi = 0$ to $\phi = \sigma$, the symmetry-breaking phase. In this era the vacuum energy density $\rho_v = V(0) = \frac{1}{4}B\sigma^4$ dominates. This large energy density causes the universe to expand exponentially. This exponential expansion continues till the temperature of the universe drops to a value T_c, after which the probability of tunneling through the potential barrier becomes significant. The value of T_c is chosen to be around 10^8 GeV (Albrecht and Steinhardt, 1982; Albrecht *et al.* 1982). Near this temperature the thermal fluctuations drive different regions of the universe away from the $SU(5)$ symmetric phase and one may expect the size of a typical fluctuation (a bubble) region to be $O(T_c^{-1})$ and the value of ϕ inside the bubble to be $O(T_c)$. These bubbles, once created, blow up rapidly, converting the whole universe to the stable phase $\phi \approx \sigma$.

We now calculate the evolution of the Higgs field ϕ from $\phi = \phi_0$, the value of ϕ after barrier penetration, to $\phi \approx \sigma$ and from this time evolution we shall determine the time required after the tunneling event for the Higgs field to roll down to the global minimum $\phi \approx \sigma$. Neglecting the loss of energy through radiation of particles, the time evolution equation of ϕ (Albrecht *et al.*, 1982) is

$$\frac{d}{dt}\left[\frac{15}{4}\dot{\phi}^2 + V(\phi)\right] = -\frac{45}{2\dot{\phi}^2}\frac{\dot{R}}{R} \tag{3.3}$$

The above equation with equation (3.2) reduces to

$$\ddot{\phi} + 3\frac{\dot{R}}{R}\dot{\phi} + \left(\frac{15}{12}g^2T^2\right)\phi - \frac{750}{128\pi^2}g^4 \ln\left(\frac{M_x}{T}\right)\phi^3 = 0 \tag{3.4}$$

Using (1.3), one can write equation (3.4) as

$$\frac{d^2\phi}{dt^2} + \frac{a}{t}\frac{d\phi}{dt} + b\phi + c\phi^3 = 0 \tag{3.5}$$

where $a = 3(\dot{R}/R)t = 3(\omega + 1/2)$, $b = (15/12)g^2T^2$, and

$$c = -\frac{750g^4}{128\pi^2}\ln\left(\frac{M_x}{T}\right) \tag{3.6}$$

Equation (3.5) will be solved in the range $\phi_0 < \phi < \sigma$ where the potential is very flat, for three different approximations, each of which has a simple analytic solution.

Case 1. The slow-rolling regime, where the field rolls at terminal velocity and the $\ddot{\phi}$ term is negligible. Under this approximation, equation (3.5) becomes

$$\frac{a}{t}\frac{d\phi}{dt} + b\phi + c\phi^3 = 0 \tag{3.7}$$

Solving, we find

$$\phi = \left(\frac{b}{c}\right)^{1/2} \frac{\phi_0}{[(\phi_0^2 + b/c)e^{(b/a)t^2} - \phi_0^2]^{1/2}} \tag{3.8}$$

Immediately after tunneling, the value of ϕ is chosen to be $\phi(0) = \phi_0 = 3 \times 10^8$ GeV, $\dot{\phi}(0) = 0$, and we neglect the finite-temperature correction. We begin our calculation at temperature $T_* \sim 10^{14}$ GeV and corresponding time $t_* \sim 10^{-14}$ GeV^{-1}. Now, the time required for the evolution of ϕ from $\phi = \phi_0$ to $\phi = \sigma$ is given by

$$\begin{aligned} t_r &= \left(\frac{a}{b}\right)^{1/2} \left\{ \log\left[\frac{\sigma^2 + q^2}{\sigma^2} \frac{\phi_0^2}{\phi_0^2 + q^2}\right]\right\}^{1/2} \\ &\simeq T_c^{-1} \sim 10^{-8}\ \text{GeV}^{-1} \end{aligned} \tag{3.9}$$

where

$$q^2 = b/c \tag{3.10}$$

Case 2. Neglecting $\ddot{\phi}$ and ϕ, since the potential is very flat in the range of interest $\phi_0 < \phi < \sigma$, equation (3.5) yields

$$\frac{a}{t}\frac{d\phi}{dt} = -c\phi^3$$

This gives, on integration,

$$\phi = \left(\frac{1}{\phi_0^2} + \frac{c}{a}t^2\right)^{-1/2} \tag{3.11}$$

and the approximate rollover time t_r is given by

$$t_r \simeq \left(\frac{a}{b}\right)\frac{1}{\phi_0} \sim 10^{-7}\ \text{GeV}^{-1} \tag{3.12}$$

Case 3. Keeping the $\ddot{\phi}$ term, i.e., assuming *a* oscillation in ϕ but that this oscillation is not too large and dropping the ϕ^3 term from equation (3.5), we get

$$\ddot{\phi} + \frac{a}{t}\frac{d\phi}{dt} + b\phi = 0 \tag{3.13}$$

The solution of ϕ for this case is given by

$$\phi = t^{-(a-1)/2}[AJ_\mu(b^{1/2}t)]$$

where $\mu = \frac{1}{2}(a-1)$ and J is the Bessel function. We have $A \sim 1$, obtained from the boundary condition. The corresponding rollover time t_r obtained by replacing J_μ by its value for large argument is given by

$$t_r \sim 10^{-8}\ \text{GeV}^{-1}$$

The calculated rollover time under the three different approximations is quite consistent. But the evolution of ϕ is not the same for the three cases.

Having determined the time evolution of ϕ and the rollover times, we now can calculate the amount of inflation occurring during this period. The value of the Brans-Dicke scalar field (ϕ_*) at the end of the Planck era is given by the general field equations (Pollock, 1982)

$$R_0^0 - \tfrac{1}{2}R \approx 8\pi\phi_*^{-1}\rho_v \leq (l_{Pl})^{-2}$$

where l_{Pl} is the Planck length. Since $(l_{Pl})^{-2} \simeq 1/G \simeq \phi_*$, we get

$$\phi_* \gtrsim (8\pi\rho_v)^{1/2}$$

The corresponding time is

$$t_* \sim \left\{\left[\frac{(5+6\omega)(3+2\omega)}{32\pi}\right]^{1/2}(8\pi)^{1/4}\rho_v^{-1/4} \sim 10^{-14}\,\text{GeV}^{-1}\right.$$

and the Planck temperature is $T_* \sim \sigma = 10^{14}$ GeV.

The calculated values of T_* and t_* are quite consistent with the initial values which we assumed at the start of our calculations. Since the theoretical rollover time is approximately 10^{-8} to $10^{-7}\,\text{GeV}^{-1}$, the amount of inflation is approximately $\sim\{10^{-8}/(4\times 10^{-13})\}^{\omega+1/2} \sim 10^{46}$, which is more than enough to explain cosmological facts (Guth, 1981). The value of ϕ after the inflation is obtained from (1.3),

$$\phi_f \sim 10^{40}\text{-}10^{42}\,\text{GeV}^2$$

or

$$G_f \sim \frac{1}{\phi_f} \sim 10^{-40}\text{-}10^{-42}\,\text{GeV}^{-2}$$

This value is quite close to the present value of $G \sim 10^{-38}\,\text{GeV}^{-2}$, despite many approximations involved in this calculation.

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